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An analytical guidance law of planetary landing mission by minimizing the control effort expenditure †

Hamed Hossein Afshari*, Alireza Basohbat Novinzadeh and Jafar Roshanian

Department of Aerospace Engineering, K.N. Toosi University of Technology, Tehran, Iran

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Abstract

An optimal trajectory design of a module for the planetary landing problem is achieved by minimizing the control effort expenditure. Using the calculus of variations theorem, the control variable is expressed as a function of costate variables, and the problem is converted into a two-point boundary-value problem. To solve this problem, the performance measure is approximated by employing a trigonometric series and subsequently, the optimal control and state trajectories are determined. To validate the accuracy of the proposed solution, a numerical method of the steepest descent is utilized. The main objective of this paper is to present a novel analytic guidance law of the planetary landing mission by optimizing the control effort expenditure. Finally, an example of a lunar landing mission is demonstrated to examine the results of this solution in practical situations.

Keywords: Nonlinear optimal control; Analytic solution; Control effort expenditure

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1. Introduction

Researchers and engineers have not been as successful in dealing with nonlinear optimal control problems as they have been in solving linear optimization problems in control. In general, the optimal formulations of nonlinear dynamic systems, either through dynamic programming or through a variational approach, lead to nonlinear partial differential equations. The numerical solution of these equations when dealing with complex nonlinear systems is always difficult, particularly for real-world physical problems. One optimal control solution of the nonlinear lunar landing mission is obtained either by a dynamic programming approach or through its variational formulation [1, 2]. Useful mathematical methods specially for approximating the mathematical functions are presented in [3]. To create a closed-loop guidance policy of the satellite injection problem, Pourtakdoust and Novinzadeh presented a fuzzy algorithm that was augmented to the solution of the time-optimal guidance strategy [4]. Afshari et al. presented some analytic approaches in spacecraft guidance [5-7]. An optimal guidance law that minimized the commanded acceleration in three dimensions was obtained by Souza [8]. Ramana has designed an optimal trajectory for soft landing on the moon by solving the boundary value equations through a numerical approach named controlled random search [9]. Lee investigated on the optimal trajectory and the feedback linearization control of a re-entry vehicle during the terminal-area energy management (TAEM) phase [10]. Employing the nonlinear function approach, an improved model-based predictive control of vehicle trajectory has been developed [11]. This paper focuses on a novel solution to design the optimal control for the nonlinear problem of planetary landing mission by optimizing the control effort expenditure. To obtain an analytical solution, a set of state-dependent nondimensional variables is introduced.

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*Corresponding author. Tel.: +98 93 6669 1025, Fax.: +98 21 77991045
E-mail address: h.h.afshari@gmail.com

Then, by using the calculus of variations theory and applying a trigonometric approximation, the performance measure is expressed with respect to the control variable. Transferring the state equations to the new system with respect to the control variable allows for analytically deriving the optimal control and optimal state trajectories through optimality relations and some boundary conditions.

2. Analytical solution for the nonlinear problem of the planetary landing mission

Consider an ideal point-mass module at the orbital of inertial frame (x, y) at $t = 0$, which moves under the action of a constant propulsive force that makes a control angle $\beta(t)$ with the horizon. Obviously, the position and velocity vector of the vehicle will change due to the action of forces upon it. The problem is to determine the optimal control strategy of this system for landing from a planet orbital by minimizing the control effort expenditure. Based on Fig. 1, the governing state-space equations are

$$\begin{cases} \dfrac{du}{dt} = -a\cos\beta, \\ \dfrac{dv}{dt} = a\sin\beta - g, \\ \dfrac{dy}{dt} = v \end{cases} \tag{1}$$

with the following appropriate boundary conditions:

$$u(t=0) = u_0,\ v(t=0) = 0,\ y(t=0) = h, \tag{2}$$

$$u(t=t_f) = 0,\ v(t=t_f) = 0,\ y(t=t_f) = 0. \tag{3}$$

For better physical understanding and for reaching an analytical explicit solution, the governing equations and the associated boundary conditions are nondimensionalized by using the following set of assumed reference parameters (u^*, v^*, y^*):

$$\bar{u} = \frac{u}{U^*}, \bar{v} = \frac{v}{U^*}, \bar{y} = \frac{y}{y^*}, \tau = \frac{t}{t^*}, \frac{d}{dt} = \frac{1}{t^*}\frac{d}{d\tau}, \tag{4}$$

whereas the reference parameters are

$$u^* = u_0, \quad y^* = h, \quad t^* = h/u_0. \tag{5}$$

Using the aforementioned nondimensional state variable Eq. (4), the transformed equations become

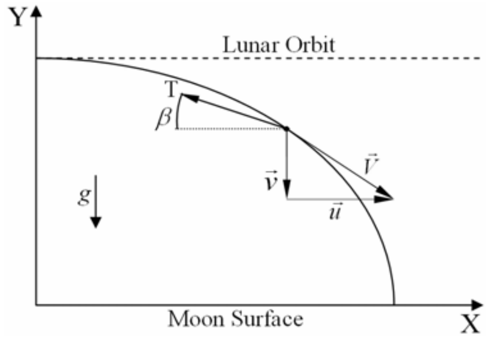

Fig. 1. Geometry of the planetary landing mission.

$$\begin{cases} \dfrac{d\bar{u}}{d\tau} = -w_1\cos\beta, \\ \dfrac{d\bar{v}}{d\tau} = w_1\sin\beta - w_2, \\ \dfrac{d\bar{y}}{d\tau} = w_3\bar{v}. \end{cases} \tag{6}$$

where

$$w_1 = \frac{a\,t^*}{u^*}, \quad w_2 = \frac{g\,t^*}{u^*}, \quad w_3 = \frac{u^*\,t^*}{y^*}. \tag{7}$$

with the following nondimensional boundary conditions:

$$\bar{u}(\tau=0) = 1, \quad \bar{v}(\tau=0) = 0, \quad \bar{y}(\tau=0) = 1, \tag{8}$$

$$\bar{u}(\tau=\tau_f) = 0, \quad \bar{v}(\tau=\tau_f) = 0, \quad \bar{y}(\tau=\tau_f) = 0. \tag{9}$$

The problem is to determine the control action of $\beta = \beta(t)$, which is required to minimize the control effort expenditure; thus, the performance measure is defined as

$$J = \int_0^{\tau_f} \beta^2 d\tau, \tag{10}$$

and the corresponding Hamiltonian is

$$H = \beta^2 - \lambda_u w_1\cos\beta + \lambda_v(w_1\sin\beta - w_2) + \lambda_y w_3\bar{v}. \tag{11}$$

Using the costate equations, the following relations are derived:

$$\begin{aligned} \lambda_u &= k_1, \\ \lambda_v &= -k_3 w_3\tau + k_2, \\ \lambda_y &= k_3. \end{aligned} \tag{12}$$

where k_1, k_2, and k_3 are constant parameters that should be determined. Note that the time derivatives in Eq. (6) can be written with respect to β. This way, β now becomes an independent variable. In addition, the boundary conditions should be expressed with respect to β. Due to the simpler form of the new state equations, it is integrated to yield the result as a function of the control variable β. This way, to keep some unsolvable integrals from appearing due to the use of the performance measure, a trigonometric series is applied to approximate the term β^2. This series is given by Dwight in [3] in the following form:

$$\beta^2 \approx \frac{\pi^2}{4} - \frac{8}{\pi}\left(\sum_{n=1}^{r} \frac{\cos((2n+1)\beta)}{(2n+1)^3}\right) = \frac{\pi^2}{4} - \frac{8}{\pi}(\mathrm{Sum}(\beta,r)) \tag{13}$$

The comparison between the real curve and its approximation when $r = 6$ is depicted in Fig. 2. Using this approximation, the costate equations remain in the previous formulation, but the optimality relation is varied. Now, the new Hamiltonian function is defined as

$$H = \frac{\pi^2}{4} - \frac{8}{\pi}(\mathrm{Sum}(\beta,r)) + \lambda_u w_1 \cos\beta + \lambda_v (w_1 \sin\beta - w_2) + \lambda_y w_3 \bar{v} . \tag{14}$$

Using the optimality relation, the optimal thrust angle can be derived as

$$-\frac{8}{\pi}\frac{d(\mathrm{Sum}(\beta,r))}{d\beta} - \lambda_u w_1 \sin\beta + \lambda_v w_1 \cos\beta = 0. \tag{15}$$

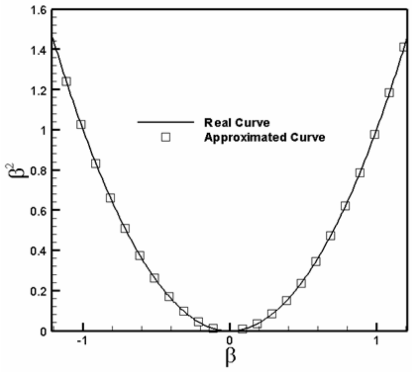

Fig. 2. Comparison of the performance measure with the produced approximation profile.

To derive an explicit formulation that can relate the thrust angle β to time τ, it will be necessary to divide Eq. (15) with respect to $\cos\beta$ as follows:

$$\frac{8}{\pi\cos\beta}\frac{d(\mathrm{Sum}(\beta,r))}{d\beta} + \lambda_u w_1 \tan\beta - \lambda_v w_1 = 0 . \tag{16}$$

By differentiating from Eq. (16) and considering Eq. (12), the required explicit relation $(d\beta/dt)$ is obtained in the following formulation:

$$\frac{d\beta}{d\tau} = \frac{\lambda_y}{\frac{8\sin\beta}{\pi}\frac{d(\mathrm{Sum}(\beta,r))}{d\beta} + \frac{8\cos\beta}{\pi}\frac{d^2(\mathrm{Sum}(\beta,r))}{d\beta^2} + \lambda_u} . \tag{17}$$

To complete the analytic solution, the time-derivative equations in the state-space equations must be written with respect to β as follows:

$$\frac{d\bar{u}}{d\beta} = \frac{8w_1\cos\beta\sin\beta}{\pi\lambda_y}\frac{d(\mathrm{Sum}(\beta,r))}{d\beta} + \frac{8w_1\cos^2\beta}{\pi\lambda_y}\frac{d^2(\mathrm{Sum}(\beta,r))}{d\beta^2} + \frac{w_1\lambda_u\cos\beta}{\lambda_y} \tag{18}$$

$$\frac{d\bar{v}}{d\beta} = \frac{w_1\sin\beta - w_2}{\lambda_y}\left(\frac{8\sin\beta}{\pi}\frac{d(\mathrm{Sum}(\beta,r))}{d\beta} + \frac{8\cos\beta}{\pi}\frac{d^2(\mathrm{Sum}(\beta,r))}{d\beta^2} + \lambda_u\right) \tag{19}$$

$$\frac{d\bar{y}}{d\beta} = \frac{8w_1 v\sin\beta}{\pi\lambda_y}\frac{d(\mathrm{Sum}(\beta,r))}{d\beta} + \frac{8w_1 v\cos\beta}{\pi\lambda_y}\frac{d^2(\mathrm{Sum}(\beta,r))}{d\beta^2} + \frac{\lambda_u w_1 v}{\lambda_y} . \tag{20}$$

With regard to the term $Sries(\beta,r)$, which is a function of β, and λ_u and λ_y, which are two constant parameter yields, it is possible to integrate Eqs. (18)-(20) while applying the following nondimensional initial conditions to the problem:

$$\bar{u}(\beta_0) = 1, \bar{v}(\beta_0) = 0, \bar{y}(\beta_0) = 1 . \tag{21}$$

Obviously, for explicit results, it is necessary to specify the magnitude of the six unknown parameters that appeared in the solution process, that is, $k_1, k_2, k_3, \beta_0, \beta_f, \tau_f$. Certainly, it can be accomplished by considering the five boundary relations illustrated from the geometry of problem and by using another relation derived from the orthogonality condition. These relations yield to six nonlinear algebraic equations abbreviated as follows:

$$\bar{u}(\beta_f) = 0,\ \bar{v}(\beta_f) = 0,\ \bar{y}(\beta_f) = 0, \\ \beta(\tau_0) = \beta_0,\ \beta(\tau_f) = \beta_f,\ H(\tau_f) = 0 . \tag{22}$$

Given $r = 6$, it is possible to find these parameters to approximate the performance measure and solve the aforementioned nonlinear algebraic equations. For a set of assumed values of the parameters u_0, h, a, g , the six unknown parameters are computed from the equations in (22). For example, in the case of a lunar landing mission, if $u_0 = h = a = 1$ and $g = 1/3$, the six unknown parameters are computed by solving the algebraic equations as

$$\tau_f = 2.2189,\ \beta_0 = -1.2125 \text{ rad},\ \beta_f = 1.3396 \text{ rad}, \\ k_1 = -1.9380,\ k_2 = 1.6928,\ k_3 = 2.3383. \tag{23}$$

3. Results and discussion

By substituting the obtained parameters of Eq. (23) in Eq. (15), the optimal thrust angle history is determined as shown in Fig. 3. In addition, considering that the solution parameters $\bar{u}, \bar{v}, \bar{y}$, and $\bar{x}$ have been expressed in the terms of β and β is related to τ, the solution parameters can be expressed with respect to τ. The time histories of the optimal state variables are depicted in Figs. 4-7. To examine the accuracy of the analytical results, a numerical method such as the steepest descent should be utilized, which is an iterative numerical method for solving the nonlinear two-point boundary-value problems. In this method, an initial guess is used to obtain the solution to a problem in which one or more of the optimality necessary conditions are not satisfied. This solution is then used to adjust the initial guess in an attempt to make the next solution come "closer" to satisfying all of the necessary conditions. If the steps are repeated and the iterative procedure converges, the optimality necessary conditions will eventually be satisfied [2]. After comparing the analytical and numerical results, the good agreements between these two different approaches are concluded. Presenting a novel analytical solution that allows some related studies such as hardware-in-the loop analysis to be performed with high reliability is the main advantage of this work. Ramana proposed a numerical technique of a controlled random search to solve a similar problem [9], but this numerical method may encounter some practical problems. In another study [8], to minimize the control effort expenditure, the commanded acceleration is introduced as the performance measure. However, according to the optimal control theory [1] and as calculated in this paper, the best measure for minimizing the control effort is

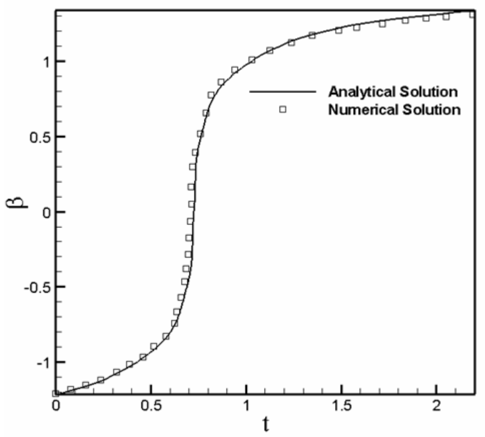

Fig. 3. Time history of the optimal control by minimizing the control effort expenditure.

$$J(u) = \int_{t_0}^{t_f} \left[\sum_{i=1}^{m} r_i u_i^2(t) \right] dt \quad or \quad J(u) = \int_{t_0}^{t_f} u^2 dt. \tag{24}$$

To demonstrate some practical applications of this solution in real-world situations, the results of several lunar landing missions with different initial conditions are presented in Tables 1 and 2. By comparing the data in these tables, it can be illustrated that by decreasing the initial altitude of the module, its control effort expenditure is reduced. However, by decreasing the initial value of the horizontal velocity component, this expenditure is increased. Furthermore, the optimal profiles of the state trajectories are depicted in Figs. 4-6. The profile of the module's downrange, as presented in Fig. 7, can be computed by integrating the horizontal velocity component with respect to the thrust angle.

Table 1. Results of some lunar landing missions with different initial altitudes.

(h,U_0)	(1,1)	(0.75,1)	(0.5,1)
τ_f	2.2189	1.9534	1.6531
β_0	-1.2125	-1.1506	-1.0309
β_f	1.3396	1.3067	1.2447
expenditure	1.3956	1.2515	1.0080

Table 2. Results of some lunar landing missions with different initial velocities.

(h,U_0)	(1,1)	(1,0.75)	(1, 0.5)
τ_f	2.2189	2.1773	2.1549
β_0	-1.2125	-1.2940	-1.3589
β_f	1.3396	1.3877	1.4306
expenditure	1.3956	1.6128	1.8121

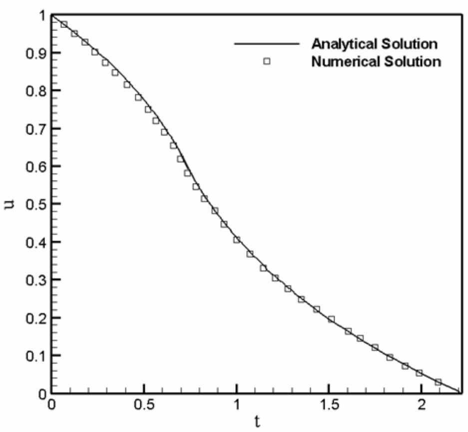

Fig. 4. Optimal trajectory of the module's horizontal velocity component.

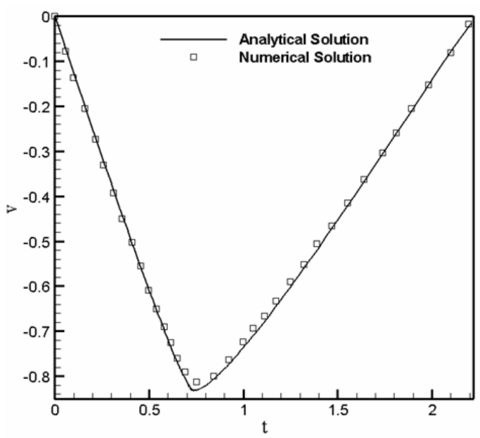

Fig. 5. Optimal trajectory of the module's vertical velocity component.

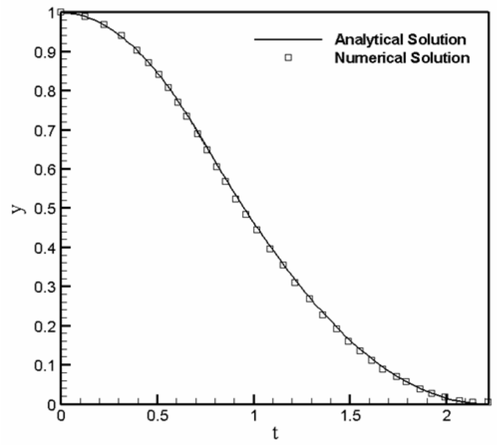

Fig. 6. Optimal trajectory of the module's altitude.

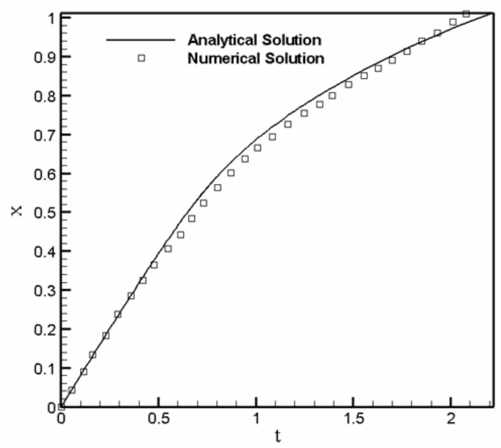

Fig. 7. Optimal trajectory of the module's downrange.

4. Conclusion

An analytical optimal control law required to influence the nonlinear problem of soft planetary landing mission is achieved by minimizing the control effort expenditure. Using an analytical solution procedure to the variational formulation yielded to the derivation of the optimal state trajectories, which can appropriately satisfy all boundary constraints. Consequently, in the proposed methodology, several difficulties associated with the numerical determination of the optimal control solution for nonlinear systems, such as a slow convergence rate, an unexpected singularity, and a high sensitivity to initial guesstimates are not observed. A conventional numerical method for solving the nonlinear optimal control problem, which is called the steepest descent utilized to validate the analytical solution. The results show that there are good agreements between the numerical and analytical results. The other advantage of this law, with respect to the previous ones, is that it is simple, easily mechanized, and can operate online in real-world spacecraft applications. By investigating the results obtained for different initial conditions, it can be concluded that by decreasing the landing initial altitude, the control effort expenditure is reduced. On the contrary, by decreasing the initial value of the horizontal velocity component, this expenditure is considerably increased.

Nomenclature

a	: Spacecraft acceleration
g	: Moon gravitational constant
h	: Height of moon orbital
u	: Velocity horizontal component
u^*, y^*, t^*	: Reference parameters
$\overline{u}, \overline{v}, \overline{y}$	: Nondimensional state variables
v	: Velocity vertical component
w_i	: Non-dimensional parameters
$x(t)$	: Spacecraft downrange
$y(t)$	: Spacecraft altitude
K_i	: Unknown constant parameters
H	: Hamiltonian function
J	: Performance measure
T	: Thrust force
β	: Thrust angle
λ_i	: Costate multiplier of optimality
τ	: Nondimensional time

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Hamed Hossein Afshari received his B.Sc. degree in the Mechanical Engineering from the K.N. Toosi University of Technology, Iran in 2006. Currently, he is studying in the M.Sc. program of the Flight Dynamics and Control course at K.N. Toosi University of Technology. His research focuses on the Control Theorem and its applications, Flight Dynamics and Guidance of aerospace systems, modeling and simulations of mechatronic systems.